Phong Normalization Factor derivation

I'll do pure-specular only (i.e. $C_d = 0$, $C_s = 1$), the mixed case is easy from there. Also, we're only interested in the maximum of reflected energy, which in the Phong model occurs when L and N are parallel to each other, which makes $R = N$ too (in all other cases, R is "on the other side" of N relative to V, hence the angle between R and V can never be smaller than the angle between R and N). Anyway, this means that $R \cdot V = N \cdot V$, which is a value we already know, namely $\cos\theta$.

Moving on, the integral we now need to calculate is

$$\int_\Omega (\cos\theta)^n \mathrm{d}\omega \tag{1}$$

with Ω being the upper hemisphere; integrating in spherical coordinates, this is

$$\int_0^{2\pi} \int_0^{\pi/2} (\cos\theta)^n \sin\theta \,\mathrm{d}\theta \mathrm{d}\phi = 2\pi \int_0^{\pi/2} (\cos\theta)^n \sin\theta \,\mathrm{d}\theta =: 2\pi I_n \tag{2}$$

and using integration by parts with $f = (\cos\theta)^n$, $g' = \sin\theta$ on I_n we get

$$\begin{aligned} I_n &= [(\cos\theta)^n(-\cos\theta)]_0^{\pi/2} - \int_0^{\pi/2} n(\cos\theta)^{n-1}(-\sin\theta)(-\cos\theta)\,\mathrm{d}\theta \\ &= \left[-(\cos\theta)^{n+1}\right]_0^{\pi/2} - n\int_0^{\pi/2} (\cos\theta)^n \sin\theta\,\mathrm{d}\theta \\ &= (-0+1) - nI_n \end{aligned}$$

so $(n+1)I_n = 1$ which means that $I_n = \frac{1}{n+1}$. Plugging this into (2) tells us that (1) equals $\frac{2\pi}{n+1}$, so the normalization factor if we want it to integrate to 1 is the reciprocal, which is $\frac{n+1}{2\pi}$.

Why $\frac{n+1}{2}$ and not $n+2$? Because this is the derivation for the original Phong formulation, where the $R \cdot V$ term is not multiplied by $\cos\theta$. If you write that version of the Phong model as a BRDF, you end up with a $\cos\theta$ in the numerator to cancel out the $\cos\theta$ factor in the reflection equation. This numerator is complete nonsense physically, so the modern formulation of the Phong model removes it. Then the integral becomes

$$\int_\Omega (R \cdot V)\ \cos\theta\,\mathrm{d}\omega \overset{L=N}{=} \int_\Omega (\cos\theta)^{n+1}\,\mathrm{d}\omega$$

and our normalization factor computation cranks out $\frac{n+2}{2}$, as expected.

Blinn-Phong normalization factor

I'll again limit myself to the specular term and again assume that the maximum reflected energy occurs with $L = N$ (I have no proof for the latter though, but I do have some experimental evidence. If I find a nice proof later, I'll update this document accordingly. Anyway, with $L = N$, things get a lot simpler than the general case because L, N, V, and H all lie in the same plane and we can work exclusively with angles. Particularly, the angle θ_h between H and N is exactly half of the angle θ between V and N, and the integral we need to evaluate boils down to

$$\int_\Omega (\cos\theta_h)^n \cos\theta \,\mathrm{d}\omega = \int_\Omega (\cos(\theta/2))^n \cos\theta \,\mathrm{d}\omega$$

(I'll only do the BRDF version with the extra factor of $\cos\theta$ here). Again integrating in spherical coordinates, we get

$$\int_0^{2\pi}\int_0^{\pi/2} (\cos(\theta/2))^n \cos\theta \sin\theta \,\mathrm{d}\theta\mathrm{d}\phi = 2\pi \int_0^{\pi/2} (\cos(\theta/2))^n \cos\theta\sin\theta\,\mathrm{d}\theta \tag{3}$$

and using the half-angle formula $\cos(\theta/2) = \sqrt{\frac{1+\cos\theta}{2}}$ and the substitution $t = \cos\theta$ (which gives $\mathrm{d}t = -\sin\theta\,\mathrm{d}\theta$) we get

$$(3) = -2\pi \int_1^0 \left(\sqrt{\frac{1+t}{2}}\right)^n t\,\mathrm{d}t = 2\pi \int_0^1 \left(\frac{1+t}{2}\right)^{n/2} t\,\mathrm{d}t$$

which can be evaluated using integration by parts, this time using $f = t$ and $g' = ((1+t)/2)^{n/2}$. This yields:

$$\begin{aligned}
&2\pi\left(\left[\frac{4}{n+2}\,t\left(\frac{1+t}{2}\right)^{(n+2)/2}\right]_{t=0}^1 - \frac{4}{n+2}\int_0^1 \left(\frac{1+t}{2}\right)^{(n+2)/2}\mathrm{d}t\right)\\
=&\frac{8\pi}{n+2}\left(\left[t\left(\frac{1+t}{2}\right)^{(n+2)/2}\right]_{t=0}^1 - \int_0^1 \left(\frac{1+t}{2}\right)^{(n+2)/2}\mathrm{d}t\right)\\
=&\frac{8\pi}{n+2}\left(\left[t\left(\frac{1+t}{2}\right)^{(n+2)/2}\right]_{t=0}^1 - \frac{4}{n+4}\left[\left(\frac{1+t}{2}\right)^{(n+4)/2}\right]_{t=0}^1\right)\\
=&\frac{8\pi}{n+2}\left(1 - \frac{4}{n+4}\left(1 - \left(\frac{1}{2}\right)^{(n+4)/2}\right)\right)\\
=&\frac{8\pi\left((n+4) - 4 + 2^{-n/2}\right)}{(n+2)(n+4)}\\
=&\frac{8\pi(n+2^{-n/2})}{(n+2)(n+4)}
\end{aligned}$$

which makes the Blinn-Phong normalization factor $\frac{(n+2)(n+4)}{8\pi(2^{-n/2}+n)}$, not $\frac{n+8}{8\pi}$.