DCT-II vs. KLT/PCA

1 Stationary Markov-1 signals

Just the definition: A *stationary Markov-1 signal* is a signal (vector) x whose autocorrelation matrix has the form

$$A = (\rho^{|i-j|})_{1 \leq i,j \leq N}$$

where $0 < \rho < 1$ is the *autocorrelation coefficient*. Typical values for ρ range between 0.95 and 0.99.

2 KLT for stationary Markov-1 signals and the DCT-II

There's a closed form for the KLT basis functions for such a process (i.e. the eigenvectors of A): They are

$$(\Phi_m)_n = \sqrt{\frac{2}{N + \mu_m}} \sin\left(w_m\left(n - \frac{N+1}{2}\right) + \frac{m\pi}{2}\right), \quad 1 \leq m, n \leq N \tag{1}$$

where

$$\mu_m = \frac{1 - \rho^2}{1 - 2\rho\cos(w_m) + \rho^2}, \quad 1 \leq m \leq N \tag{2}$$

with w_m being the real roots of the equation

$$\tan(Nw) = -\frac{(1-\rho^2)\sin(w)}{\cos(w) - 2\rho + \rho^2\cos(w)} \tag{3}$$

in the interval $(0, \pi)$. Proof of this can be found in [1]. Letting $\rho \to 1$ in (3), one obtains:

$$\tan(Nw) = -\frac{0}{2(\cos(w) - 1)} \tag{4}$$

and since the real roots of tan are precisely $k\pi$ with $k \in \mathbb{Z}$, a suitable choice of w_k is

$$w_k = \frac{\pi(k-1)}{N}, \quad 1 \leq k \leq N.$$

For even k, $\cos(w_k) = -1$ and (4) is well-defined. For odd k though, $\cos(w_k) = 1$ and hence the denominator is zero; applying l'Hospitals rule yields:

$$\lim_{\omega \to w_k} -\frac{0}{2(\cos(\omega) - 1)} \overset{2\times\text{l'Hospital}}{=} \lim_{\omega \to w_k} \frac{0}{2\cos(\omega)} = 0,$$

so odd k are valid too. Plugging this into (2) yields that μ_k must be zero for $2 \le k \le N$. For $k = 1$ the denominator is zero again; using a different trick this time, we note that the main diagonal of A consists only of ones for any choice of ρ, and since the trace of a matrix is invariant under similarity transformations (change of basis), we have

$$N = \sum_{i=1}^{N} (A)_{ii} = \operatorname{tr}(A) = \operatorname{tr}(T^{-1}AT) = \operatorname{tr}(\operatorname{diag}(\mu_1, \ldots, \mu_N)) = \sum_{i=1}^{N} \mu_i = \mu_1.$$

Inserting this into (1) yields:

$$\begin{aligned}
(\Phi_m)_n &= \sqrt{\frac{2}{N + \delta_{m1} N}} \, \sin\left(\frac{(m-1)\pi}{N}\left(n - \frac{N+1}{2}\right) + \frac{m\pi}{2}\right) \\
&= \sqrt{\frac{2}{N}} \, c_m \, \sin\left(\frac{(m-1)(n-\frac{1}{2})\pi}{N} - \frac{(m-1)N\pi}{2N} + \frac{m\pi}{2}\right) \\
&= \sqrt{\frac{2}{N}} \, c_m \, \sin\left(\frac{(m-1)(n-\frac{1}{2})\pi}{N} + \frac{\pi}{2}\right) \\
&= \sqrt{\frac{2}{N}} \, c_m \, \cos\left(\frac{(m-1)(n-\frac{1}{2})\pi}{N}\right)
\end{aligned}$$

where

$$c_m = \begin{cases} \frac{1}{\sqrt{2}} & m = 1, \\ 1 & \text{otherwise,} \end{cases}$$

which are precisely the DCT-II basis functions.

Figure 1 shows the KLT basis functions with $\rho = 0.95$ and the DCT-I and DCT-II basis functions for $N = 16$.

References

[1] Ray, W.D. und Driver, R.M.: *Further decomposition of the Karhunen-Loève series representation of a stationary random process.* IEEE Transactions on Information Theory, 16(6) pp. 663–668, November 1970.

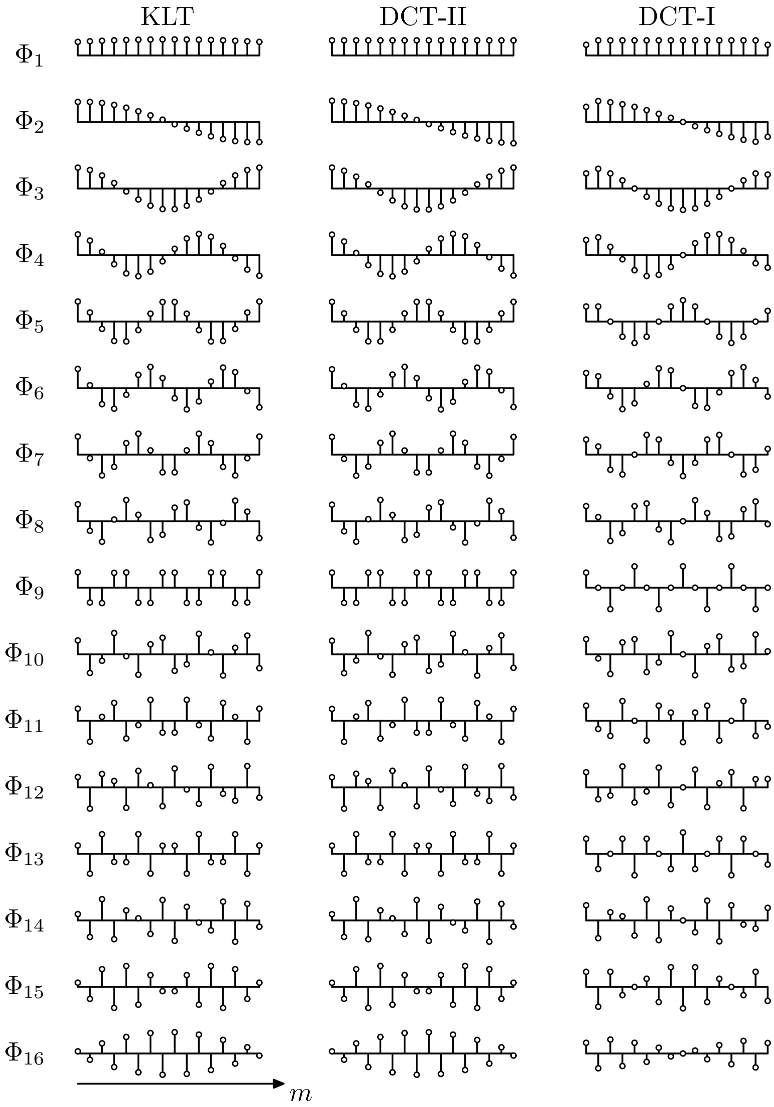

Figure 1: KLT, DCT-I and DCT-II basis functions with $N = 16$, $\rho = 0.95$.